NSW GOVERNMENT

NSW Education Standards Authority

2020 **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6–14)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the length of the vector $-\underset{\sim}{i} + 18\underset{\sim}{j} - 6\underset{\sim}{k}$?

A. 5

B. 19

C. 25

D. 361

2 Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, what are the values of p and q?

A. $p = -6, \quad q = \sqrt{10}$

B. $p = -6, \quad q = 10$

C. $p = 6, \quad q = \sqrt{10}$

D. $p = 6, \quad q = 10$

3 What is the Cartesian equation of the line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

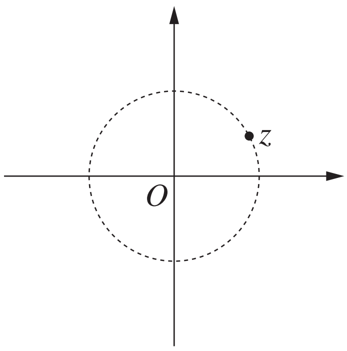
A. $2y + x = 7$

B. $y - 2x = -5$

C. $y + 2x = 5$

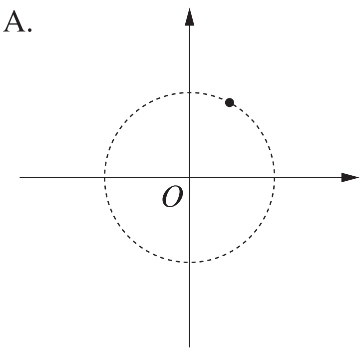
D. $2y - x = -1$

4 The diagram shows the complex number z on the Argand diagram.

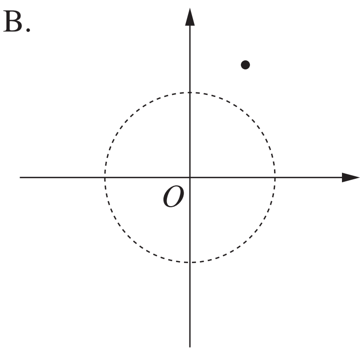


Which of the following diagrams best shows the position of $\dfrac{z^2}{|z|}$?

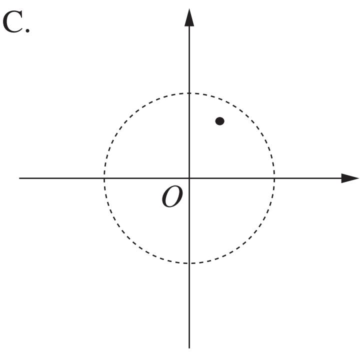
A.



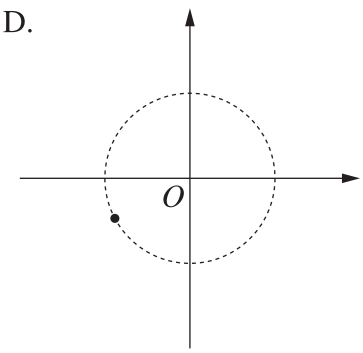
B.



C.



D.



5 A particle undergoing simple harmonic motion has a maximum acceleration of 6 m/s^2 and a maximum velocity of 4 m/s.

What is the period of the motion?

A. π

B. $\dfrac{2\pi}{3}$

C. 3π

D. $\dfrac{4\pi}{3}$

6 Which expression is equal to $\displaystyle\int \frac{1}{x^2+4x+10}\,dx$?

A. $\dfrac{1}{\sqrt{6}}\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

B. $\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

C. $\dfrac{1}{2\sqrt{6}}\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

D. $\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

7 Consider the proposition:

'If 2^n-1 is not prime, then n is not prime'.

Given that each of the following statements is true, which statement disproves the proposition?

A. 2^5-1 is prime

B. 2^6-1 is divisible by 9

C. 2^7-1 is prime

D. $2^{11}-1$ is divisible by 23

8 Consider the statement:

'If n is even, then if n is a multiple of 3, then n is a multiple of 6'.

Which of the following is the negation of this statement?

A. n is odd and n is not a multiple of 3 or 6.

B. n is even and n is a multiple of 3 but not a multiple of 6.

C. If n is even, then n is not a multiple of 3 and n is not a multiple of 6.

D. If n is odd, then if n is not a multiple of 3 then n is not a multiple of 6.

9 What is the maximum value of $\left|e^{i\theta}-2\right|+\left|e^{i\theta}+2\right|$ for $0 \le \theta \le 2\pi$?

A. $\sqrt{5}$

B. 4

C. $2\sqrt{5}$

D. 10

10 Which of the following is equal to $\int_0^{2a} f(x)\,dx$?

A. $\int_0^{a} f(x)-f(2a-x)\,dx$

B. $\int_0^{a} f(x)+f(2a-x)\,dx$

C. $2\int_0^{a} f(x-a)\,dx$

D. $\int_0^{a} \frac{1}{2}f(2x)\,dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) Consider the complex numbers $w = -1 + 4i$ and $z = 2 - i$.

(i) Evaluate $|w|$. **1**

(ii) Evaluate $w\overline{z}$. **2**

(b) Use integration by parts to evaluate $\displaystyle\int_1^e x \ln x \, dx$. **3**

(c) A particle starts at the origin with velocity 1 and acceleration given by **3**

$$a = v^2 + v,$$

where v is the velocity of the particle.

Find an expression for x, the displacement of the particle, in terms of v.

(d) Consider the two vectors $\underset{\sim}{u} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$. **3**

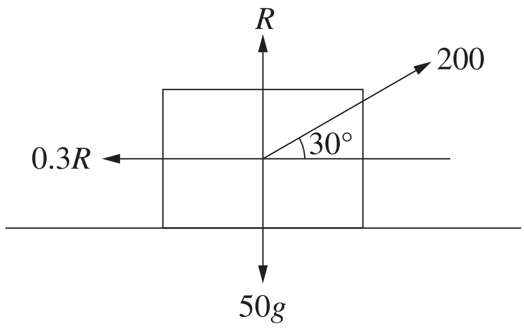
For what values of p are $\underset{\sim}{u} - \underset{\sim}{v}$ and $\underset{\sim}{u} + \underset{\sim}{v}$ perpendicular?

(e) Solve $z^2 + 3z + (3 - i) = 0$, giving your answer(s) in the form $a + bi$, where a and b are real. **4**

Question 12 (14 marks) Use the Question 12 Writing Booklet

(a) A 50-kilogram box is initially at rest. The box is pulled along the ground with a force of 200 newtons at an angle of 30° to the horizontal. The box experiences a resistive force of $0.3R$ newtons, where R is the normal force, as shown in the diagram.

Take the acceleration g due to gravity to be 10 m/s^2.



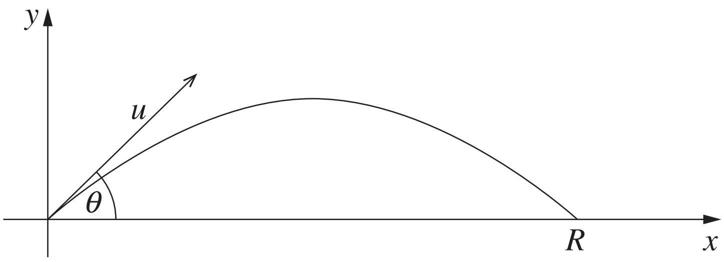
(i) By resolving the forces vertically, show that $R = 400$. **2**

(ii) Show that the net force horizontally is approximately 53.2 newtons. **2**

(iii) Find the velocity of the box after the first three seconds. **2**

Question 12 continues on page 8

(b) A particle is projected from the origin with initial velocity u m/s at an angle θ to the horizontal. The particle lands at $x = R$ on the x-axis. The acceleration vector is given by $\underset{\sim}{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$, where g is the acceleration due to gravity. (Do NOT prove this.)



(i) Show that the position vector $\underset{\sim}{r}(t)$ of the particle is given by **3**

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}.$$

(ii) Show that the Cartesian equation of the path of flight is given by **3**

$$y = \frac{-gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right).$$

(iii) Given $u^2 > gR$, prove that there are 2 distinct values of θ for which the particle will land at $x = R$. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) A particle is undergoing simple harmonic motion with period $\frac{\pi}{3}$. The central point of motion of the particle is at $x = \sqrt{3}$. When $t = 0$ the particle has its maximum displacement of $2\sqrt{3}$ from the central point of motion. **3**

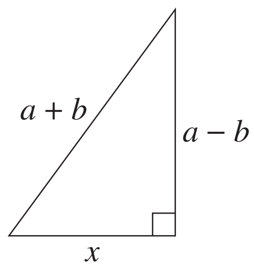
Find an equation for the displacement, x, of the particle in terms of t.

(b) Consider the two lines in three dimensions given by **3**

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

By equating components, find the point of intersection of the two lines.

(c) (i) By considering the right-angled triangle below, or otherwise, prove that $\frac{a+b}{2} \geq \sqrt{ab}$, where $a > b \geq 0$. **2**



(ii) Prove that $p^2 + 4q^2 \geq 4pq$. **1**

(d) (i) Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$. **1**

(ii) By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^4$, show that **3**

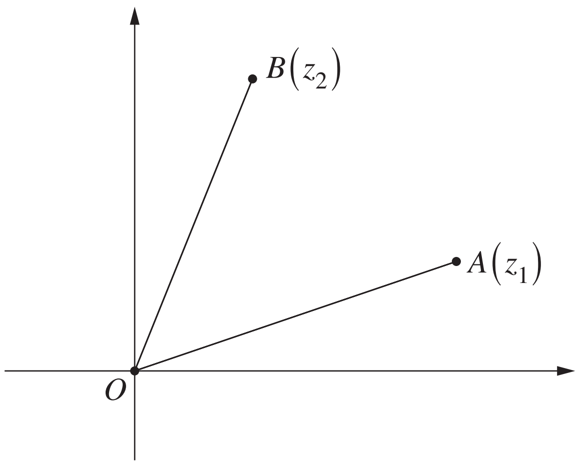
$$\cos^4\theta = \frac{1}{8}\left(\cos(4\theta) + 4\cos(2\theta) + 3\right).$$

(iii) Hence, or otherwise, find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$. **2**

Question 14 (16 marks) Use the Question 14 Writing Booklet

(a) Let z_1 be a complex number and let $z_2 = e^{\frac{i\pi}{3}} z_1$.

The diagram shows points A and B which represent z_1 and z_2, respectively, in the Argand plane.



(i) Explain why triangle OAB is an equilateral triangle. **2**

(ii) Prove that $z_1^2 + z_2^2 = z_1 z_2$. **3**

(b) A particle starts from rest and falls through a resisting medium so that its acceleration, in m/s^2, is modelled by **4**

$$a = 10\left(1 - (kv)^2\right),$$

where v is the velocity of the particle in m/s and $k = 0.01$.

Find the velocity of the particle after 5 seconds.

(c) Prove by mathematical induction that, for $n \geq 2$, **4**

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

(d) Prove that for any integer $n > 1$, $\log_n(n+1)$ is irrational. **3**

Question 15 (13 marks) Use the Question 15 Writing Booklet

(a) In the set of integers, let P be the proposition:

'If $k+1$ is divisible by 3, then k^3+1 is divisible by 3'.

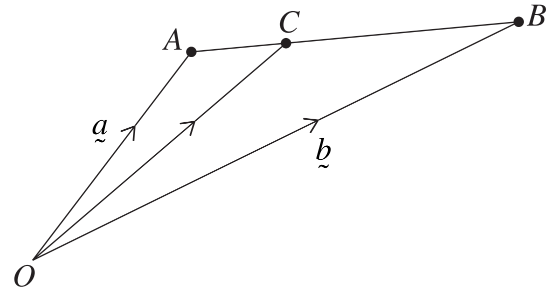
(i) Prove that the proposition P is true. **2**

(ii) Write down the contrapositive of the proposition P. **1**

(iii) Write down the converse of the proposition P and state, with reasons, whether this converse is true or false. **3**

Question 15 continues on page 12

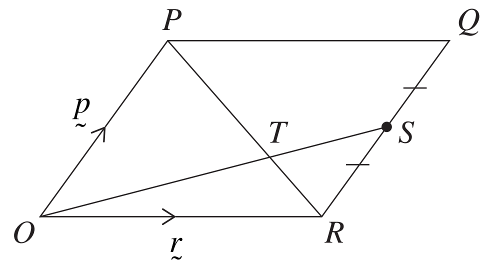
(b) The point C divides the interval AB so that $\frac{CB}{AC} = \frac{m}{n}$. The position vectors of A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, as shown in the diagram.



(i) Show that $\overrightarrow{AC} = \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$. **2**

(ii) Prove that $\overrightarrow{OC} = \frac{m}{m+n}\underset{\sim}{a} + \frac{n}{m+n}\underset{\sim}{b}$. **1**

Let $OPQR$ be a parallelogram with $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OR} = \underset{\sim}{r}$. The point S is the midpoint of QR and T is the intersection of PR and OS, as shown in the diagram.



(iii) Show that $\overrightarrow{OT} = \frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$. **3**

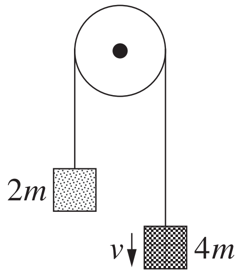
(iv) Using parts (ii) and (iii), or otherwise, prove that T is the point that divides the interval PR in the ratio $2:1$. **1**

End of Question 15

Question 16 (16 marks) Use the Question 16 Writing Booklet

(a) Two masses, $2m$ kg and $4m$ kg, are attached by a light string. The string is placed over a smooth pulley as shown.

The two masses are at rest before being released and v is the velocity of the larger mass at time t seconds after they are released.



The force due to air resistance on each mass has magnitude kv, where k is a positive constant.

(i) Show that $\dfrac{dv}{dt} = \dfrac{gm - kv}{3m}$. **2**

(ii) Given that $v < \dfrac{gm}{k}$, show that when $t = \dfrac{3m}{k}\ln 2$, the velocity of the larger mass is $\dfrac{gm}{2k}$. **3**

Question 16 continues on page 14

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\, d\theta,\ n = 0, 1, \ldots$.

(i) Prove that $I_n = \dfrac{2n}{2n+1} I_{n-1},\ n \geq 1.$ **3**

(ii) Deduce that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}.$ **3**

Let $J_n = \int_0^1 x^n (1-x)^n\, dx,\ n = 0, 1, 2, \ldots$.

(iii) Using the result of part (ii), or otherwise, show that $J_n = \dfrac{(n!)^2}{(2n+1)!}.$ **3**

(iv) Prove that $\left(2^n n!\right)^2 \leq \left(2n+1\right)!.$ **2**

End of paper

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

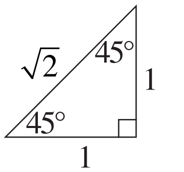
$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

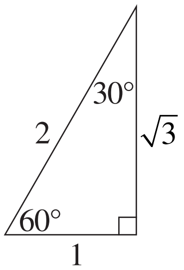
Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$



$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

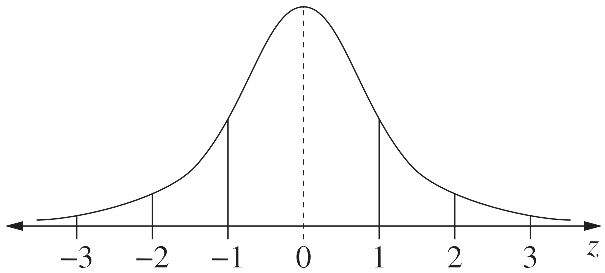
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$